On the Word Error Probability of Linear Block Codes for Diversity Systems in Mobile Communications

Chachag Yi† and Jae Hong Lee†, Nonmembers

SUMMARY The word error probability of linear block codes is computed for diversity systems with maximal ratio combining in mobile communications with three decoding algorithms: error correction (EC), error/erasure correction (EEC), and maximum likelihood (ML) soft decoding algorithm. Ideal interleaving is assumed. EEC gives 0.1–1.5 dB gain over EC. The gain of EEC over EC decreases as the number of diversity channels increases. ML soft gives 1.8–5.5 dB gain over EC. key words: linear block codes, decoding algorithms, word error probability, diversity systems, Rayleigh fading channels

1. Introduction

Mobile communication channels suffer from noise and fading due to multipath propagation [1]. Diversity systems are effective techniques for fading compensation in mobile communications. In diversity systems, one symbol is transmitted over L diversity channels and respective L signals are combined at the receiver. The performance is improved at the expense of high implementation complexity [2]. There are three types of diversity combining schemes in practical use: selection combining (SC), equal gain combining (EGC), and maximal ratio combining (MRC). In SC, the best signal among L channels is chosen and used [3]. In EGC, each signal of L channels is cophased, equally weighted, and summed to give the resultant output [4]. In MRC, each signal of L channels is cophased, weighted by the gain of each channel which is proportional to the signal to noise ratio, and summed to give the resultant output [5]. MRC yields maximal possible improvement that diversity systems can obtain.

Linear block codes are used for fading compensation in mobile communications [6], [7]. The error performance of linear block codes depends on the decoding algorithm. In error/erasure correction decoding algorithm, after declaring unreliable symbols as erasures, errors and erasures are corrected. In maximum likelihood soft decoding algorithm, a codeword is chosen which maximizes the conditional probability that the codeword was sent given the received vector [8].

In this paper, binary linear block codes are examined for diversity systems with maximal ratio combining in mobile communications. Coherent BPSK is used and ideal interleaving is assumed. The word error probability is computed in Rayleigh fading channels with three decoding algorithms: error correction (EC), error/erasure correction (EEC), and maximum likelihood (ML) soft decoding algorithms.

This paper is organized as follows. In Sect. 2 system model is described and the word error probability of binary linear block codes is analyzed for EC, EEC, and ML soft decoding algorithms. Numerical results are given in Sect. 3. In Sect. 4 conclusion is drawn.

2. Word Error Probability of Linear Block Codes for Diversity Systems

Consider digital communication systems with L diversity channels in mobile communications. Mobile communication channels are modeled as Rayleigh fading channels. In mobile communications, it is reasonable to assume that fading is so slow that the signal to noise ratio (SNR) is constant at least during symbol duration. Assume that signals in L channels undergo independent fading. Consider binary codes denoted by (n, k) with length n and dimension k. Coherent BPSK is used. The decision variable for jth code symbol in lth channel is given by

\[ r_{jl} = \sqrt{E_c} (-1)^{c_{jl}} a_{jl} e^{-j\phi_{jl}} + N_{jl}, \]

\[ j=1, \ldots, n, \quad l=1, \ldots, L, \]  

(1)

where \( E_c \) is symbol energy on each channel, \( c_{jl} \) is code symbol, \( a_{jl} \) is the attenuation factor, \( e^{-j\phi_{jl}} \) is the phase shift, and \( N_{jl} \) is a Gaussian random variable with mean zero and variance \( N_0/2 \). Random variables \( N_{jl} \) are mutually independent. Assume that each channel has identical average signal to noise ratio (SNR), i.e.,

\[ E(a_{jl}^2) = E(a^2) \]

where \( E(\cdot) \) denotes statistical averaging. The average SNR on each channel is given by

\[ \Gamma_c = E_c N_0 E(a^2). \]

(2)

Assume that maximal ratio combining (MRC) are used. When MRC is used for diversity systems, \( \Gamma_c \) is given by

\[ \Gamma_c = \frac{1}{L} \frac{k}{n} \Gamma_b, \]

(3)
where $T_b$ is average SNR per bit. The instantaneous SNR $\gamma$ after maximal ratio combining becomes a chi-square distributed random variable with $2L$ degrees of freedom [9]. The probability density function of $\gamma$ is given by

$$p(\gamma) = \frac{1}{(L-1)!! T_c} \gamma^{L-1} e^{-\gamma/2T_c}. \tag{4}$$

The symbol error probability with instantaneous SNR $\gamma$ is given by

$$P_e(\gamma) = Q(\sqrt{2}\gamma), \tag{5}$$

where $Q(x) = \int_x^\infty e^{-y^2/2} \, dy$.

Consider the case that error correction (EC) decoding algorithm is used. With ideal interleaving, error events of each symbol in a codeword are mutually independent. The average symbol error probability is given by [9]

$$P_e = \int_0^\infty p(\gamma) Q(\sqrt{2}\gamma) \, d\gamma = \frac{1}{2} \left[ 1 - \frac{1}{1 + \frac{1}{T_c}} \sum_{l=1}^{L-1} \frac{1}{l} \right]. \tag{6}$$

Let $e$ be the number of errors in a codeword. Correct decoding is guaranteed as long as $2e + s + 1 \leq d$ where $d$ is the minimum distance of code. The word error probability is given by

$$P_w = \sum_{e=0}^{n-s} \left( \begin{array}{c} n \\ e \end{array} \right) P_e^e (1 - P_e)^{n-e}, \tag{7}$$

where $\lfloor x \rfloor$ is the greatest integer that does not exceed $x$.

Consider the case that error/erasure correction (EEC) decoding algorithm is used. With ideal interleaving, error events and erasure events of each symbol in a codeword are mutually independent. Symbols having SNR below erasure threshold $T_e$ are erased. The symbol erasure probability is given by

$$P_{e|e} = \int_0^{T_e} p(\gamma) \, d\gamma = 1 - e^{-T_e/T_c} \sum_{l=0}^{L-1} \frac{T_e^l}{l!} \frac{1}{l}. \tag{8}$$

The symbol error probability among not erased symbols is given by (see appendix)

$$P_{ee} = \int_{T_e}^\infty p(\gamma) Q(\sqrt{2}\gamma) \, d\gamma = Q(\sqrt{2}T_e) e^{-T_e/T_c} \sum_{l=0}^{L-1} \frac{T_e^l}{l!} \frac{1}{l} \frac{Q(\sqrt{2}T_e) e^{-T_e/T_c} \sum_{l=0}^{L-1} \frac{T_e^l}{l!} \frac{1}{l}}{\sqrt{1 + \frac{1}{T_c}}} \frac{b(l)}{l!(1 + T_c)^{2l}} \cdot \frac{1}{\beta^{2m-1}} e^{-\beta^2/2}, \tag{9}$$

where $\beta = \sqrt{2}T_e/(1 + 1/T_c)$ and $b(l) = (2l)!/(l! 2^l)$. In (9) $Q(\cdot)$ is readily evaluated using a personal computer with the algorithm in [10]. Let $e$ and $s$ be the number of errors and erasures in a codeword, respectively. Correct decoding is guaranteed as long as $2e + s + 1 \leq d$. The word error probability is given by

$$P_w = \sum_{e=s}^{n-s} \left( \begin{array}{c} n \\ s \end{array} \right) P_{e|e}^s (1 - P_{ee})^{n-s}$$

$$+ \sum_{e=0}^{n-s} \left( \begin{array}{c} n \\ s \end{array} \right) \sum_{i=0}^{n-s} \left( \begin{array}{c} n-s \\ i \end{array} \right) P_{e|e}^i P_{ee}^{n-s-i}$$

$$- P_{e|e}^s - P_{ee}^{n-s}, \tag{10}$$

Consider the case that maximum likelihood (ML) soft decoding algorithm is used. Let $P(i)$ be the probability that a received vector is incorrectly decoded as a codeword apart Hamming distance $i$ from the transmitted codeword. $P(i)$ is just the average symbol probability of BPSK with $iL$ diversity channels [9]. Therefore, $P(i)$ is given by

$$P(i) = \frac{1}{2} \left[ 1 - \frac{1}{1 + \frac{1}{T_c}} \sum_{j=0}^{a_j} \left( \begin{array}{c} j \\ i \end{array} \right) \frac{1}{2j} \frac{1}{1 + (1 + T_c)/2j} \right]. \tag{11}$$

The word error probability is given by

$$P_w \leq \sum_{i=0}^{n} A_i P(i) \tag{12}$$

where $A_i$ is the number of codewords with Hamming weight $i$.

3. Numerical Results

The word error probability of binary linear block codes is computed for diversity systems in Rayleigh fading channels with three decoding algorithms: error correction (EC), error/erasure correction (EEC), and maximum likelihood (ML) soft decoding algorithm. Numerical results are obtained by using (7), (10), and (12).

In Fig. 1, the word error probability $P_w$ of (7,4) Hamming code is shown versus erasure threshold $T_e$ for EEC decoding algorithm. Let $T_{e, opt}$ be optimum erasure threshold which gives minimum word error probability. If $T_e$ is below $T_{e, opt}$, too many erroneous symbols are not declared as erasures so $P_w$ increases. The word error probability of EC is obtained when $T_e$ is zero. If $T_e$ is above $T_{e, opt}$, too many correct symbols are declared as erasures so $P_w$ increases. When the
Fig. 1 Word error probability of (7, 4) Hamming code versus erasure threshold for error/erasure correction decoding algorithm.

- a: \( L = 1 \), average SNR per bit = 10 dB
- b: \( L = 2 \), average SNR per bit = 10 dB
- c: \( L = 4 \), average SNR per bit = 10 dB
- d: \( L = 1 \), average SNR per bit = 15 dB
- e: \( L = 2 \), average SNR per bit = 15 dB
- f: \( L = 4 \), average SNR per bit = 15 dB

Fig. 2 Word error probability of (7, 4) Hamming code for error correction, error/erasure correction, and maximum likelihood soft decoding algorithms. \( L \): the number of diversity channel.

In Fig. 2, \( P_W \) of (7, 4) Hamming code is shown for EC, EEC, and ML soft decoding algorithms. When \( L = 1 \), at \( P_W = 10^{-3} \), EEC gives 1.5 dB gain over EC and ML soft gives 4.0 dB gain over EC. When \( L = 4 \), at \( P_W = 10^{-3} \), EEC gives 0.3 dB gain over EC and ML soft gives 1.8 dB gain over EC. When \( L = 4 \), at \( P_W = 10^{-6} \), EEC gives 0.3 dB gain over EC and ML soft gives 3.0 dB gain over EC. The gain of EEC over EC is not much dependent on the average SNR per bit. The gain of EEC over EC decreases as \( L \) increases. The instantaneous SNR after combining has smaller variance as \( L \) increases. EEC decoding is more effective when instantaneous SNR has large variance because it is easy to make erasures in this case. The gain of ML soft over EC increases as average SNR per bit increases.

In Fig. 3, \( P_W \) of (23, 12) Golay code is shown for EC, EEC, and ML soft decoding algorithms. When \( L = 1 \), at \( P_W = 10^{-3} \), EEC gives 1.2 dB gain over EC and ML soft gives 5.5 dB gain over EC. When \( L = 4 \), at \( P_W = 10^{-3} \), EEC gives 0.1 dB gain over EC and ML soft gives 2.3 dB gain over EC. The gains of (23, 12) Golay code among decoding algorithms are similar to those of (7, 4) Hamming code.

4. Conclusion

The word error probabilities of binary linear block codes have been presented for diversity systems in Rayleigh fading channels with error correction (EC), error/erasure correction (EEC), and maximum likelihood (ML) soft decoding algorithm. Coherent BPSK and maximal ratio combining are used and ideal interleaving is assumed. For EEC decoding algorithm, the equations of the symbol erasure probability and the average symbol error probability are derived in the closed form. The word error probabilities are readily evaluated using a personal computer for the three decoding algorithms. Optimum erasure threshold exists which gives minimum word error probability for EEC decoding algorithm. When either the number of channel or average SNR per bit increases, the probability that instantaneous SNR is low is reduced. By the way it is necessary to maintain appropriate number of erasures for EEC decoding, therefore optimum erasure threshold increases. In (7, 4) Hamming code, EEC gives 0.3–1.5 dB gain over EC and ML soft gives 1.8–
4.0 dB gain over EC. When the number of diversity channels is small, EEC decoding is more effective because it is easy to make erasures. In (23, 12) Golay code, EEC has 0.1–1.2 dB gain over EC and ML soft has 2.3–5.5 dB gain over EC. The gains of (23, 12) Golay code among decoding algorithms are similar to those of (7, 4) Hamming code.

References


Appendix: Derivation of (9)

The average symbol error probability among not erased symbols is given by

\[
P_{e|x} = \int_{T_c}^{\infty} p(\gamma) Q(\sqrt{2\gamma}) d\gamma
\]

\[
= \int_{T_c}^{\infty} \frac{1}{(L-1)!} \left( \frac{T_c}{\sigma_e^2} \right)^{-1} e^{-\gamma T_c} d\gamma
\]

\[
\cdot \int_{T_c}^{\infty} e^{-\gamma T_c} d\gamma
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\sigma_e^2 T_c}^{\infty} e^{-y^2/2} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{T_c}{\sigma_e^2 T_c} \right)^{m} e^{-y T_c} dy
\]

\[
= -\left( \frac{1}{2T_c^2} \right)^{\frac{1}{2}} e^{-\frac{1}{2}T_c^2} dy
\]

\[
= Q(\sqrt{2T_c}) e^{-\frac{1}{2}T_c^2} \sum_{m=0}^{\infty} \left( \frac{T_c}{T_c} \right)^{m} \frac{1}{m!}
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{T_c^2} \frac{1}{\sqrt{2\pi}} \frac{1}{2^m} \cdot \int_{\sigma_e^2 T_c}^{\infty} y^m e^{-\frac{1}{2}(y^2 + \frac{T_c^2}{y^2})} dy.
\]

The integration formula in the second term is written by

\[
\int_{\sigma_e^2 T_c}^{\infty} gy^m e^{-\frac{1}{2}(y^2 + \frac{T_c^2}{y^2})} dy
\]

\[
= \frac{1}{(1 + \frac{T_c}{g})} \frac{1}{2^m} \int_{\sigma_e^2 T_c}^{\infty} s^{2m} e^{-s^{2}/2} ds.
\]

where \( \beta = \sqrt{2T_c} \left( 1 + \frac{1}{T_c} \right) \). Using integration by parts \( l \) times, (A.3) is written by

\[
\int_{\sigma_e^2 T_c}^{\infty} gy^m e^{-\frac{1}{2}(y^2 + \frac{T_c^2}{y^2})} dy
\]

\[
= \frac{1}{(1 + \frac{T_c}{g})} \frac{1}{2^m} \int_{\sigma_e^2 T_c}^{\infty} s^{2m-1} e^{-s^{2}/2} ds
\]

\[
= \frac{1}{(1 + \frac{T_c}{g})} \frac{1}{2^m} \left[ \beta^{2m-1} + (2l-1) \beta^{2l-3} + \ldots + (2l-1) (2l-3) \ldots \beta \right] e^{-s^{2}/2}
\]

\[
+ (2l-1) (2l-3) \ldots \beta \int_{\sigma_e^2 T_c}^{\infty} e^{-s^{2}/2} ds
\]

\[
= \frac{1}{(1 + \frac{T_c}{g})} \frac{1}{2^m} \left[ \frac{e^{-s^{2}/2}}{s^{2m-1}} \right]_{s=\infty}^{s=\beta}
\]

\[
+ b(l) \frac{b(l)}{b(m)}
\]

where \( b(l) = (2l)! / (l! 2^l) \). From (A.2) and (A.3) \( P_{e|x} \) is given by

\[
P_{e|x} = Q\left( \sqrt{2T_c} \right) e^{-\frac{1}{2}T_c^2} \sum_{m=0}^{\infty} \left( \frac{T_c}{T_c} \right)^{m} \frac{1}{m!} \frac{Q(\beta)}{\sqrt{1 + \frac{T_c}{g}}}
\]

\[
\cdot \left( \frac{1}{2T_c^2} \right)^{\frac{1}{2}} e^{-\frac{1}{2}T_c^2} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{T_c^2} \frac{1}{\sqrt{2\pi}} \frac{1}{2^m} \int_{\sigma_e^2 T_c}^{\infty} y^m e^{-\frac{1}{2}(y^2 + \frac{T_c^2}{y^2})} dy
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{T_c^2} \frac{1}{\sqrt{2\pi}} \frac{1}{2^m} \sum_{l=0}^{\infty} \frac{b(l)}{b(m)}
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{T_c^2} \frac{1}{\sqrt{2\pi}} \frac{1}{2^m} \sum_{l=0}^{\infty} \frac{b(l)}{b(m)}
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{T_c^2} \frac{1}{\sqrt{2\pi}} \frac{1}{2^m} \sum_{l=0}^{\infty} \frac{b(l)}{b(m)}
\]